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A CLASS OF GENERAL RELIABILITY GROWTH PREDICTION MODELS

by

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ABSTRACT:

In viewing reliability growth prediction models as Markov chains, it is seen that the computation of the reliability after n trials and possible associated repairs, R_n , may be accomplished with any of several different methods. A class of models is considered which accommodates variations in several important factors such as the interdependencies of assignable cause failure modes, inclusion of an inherent failure mode, the repair policy, and the distribution of initial states of the system.

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1. Introduction

A problem which has developed into one of considerable importance and interest is that of accounting for the improvement or upgrading of reliability through corrective action on an item or system. This problem arises in the development stage when failures are relatively frequent and various failure modes, observed through design inspection or testing, can be removed. Whether corrective action takes place through design changes or development modifications, it is common to assume that the action never decreases reliability and, at least occasionally, actually increases reliability, hence the term reliability growth. Generally speaking, the mathematical models developed for this problem concern two objectives: (1) the prediction, before testing is initiated, of the reliability that should be attained at various stages of development; and (2) the estimation of reliability as a function of one or more parameters from test data. In general, the latter objective is attained by making statistical inferences from observed test data, while the former is gained through the use of probabilistic models using certain assumptions concerning the failure modes and the repair policy.

This paper concerns the exposition, in a uniform setting, of several prediction models appearing in the literature and incorporation of some general features into them. We therefore begin with a brief review of these models.

Lloyd and Lipow [2] consider a system assumed to contain a single failure mode. A testing program is carried out for the purpose of discovering and removing the failure mode. On each trial, the system fails with (known) probability q (i.e., the system has reliability $1 - q$) if the failure mode has not been corrected; otherwise, it has reliability 1. On each trial resulting in a failure, an attempt is made to permanently repair the failure mode. It is assumed that on each such attempt the probability ϕ of succeeding is known.

Thus, in the model of Lloyd and Lipow, the system is either in an unrepaired state or a repaired state after n tests have been performed. Using a difference equation for the probability that the device is in the unrepaired state after n tests, Lloyd and Lipow show that the reliability of the system for the $n + 1^{st}$ test is $1 - q(1 - q\phi)^n$.

Pollock [3] considers a similar system, in which the repair probability ϕ is known, but the failure rate q' of the "repaired" system is not necessarily zero (nor even less than q). Under various assumptions concerning the distribution of failure occurrences (such as Poisson or Bernoulli) and whether q and q' are known or have some known prior distribution, Pollock obtains the failure rate distributions at time τ (in the continuous case) and the density of the failure probability at the $n + 1^{st}$ trial (in the discrete case).

Weiss [5] considers a system possibly containing several failure modes, each with the same exponential failure distribution. It is assumed

that the failure modes present in the system are drawn from a population of potential failure modes. This allows some relief from the rather strict assumption that all failure modes have identically the same failure rate, in that a higher failure rate for a mode can be modeled roughly by a higher probability of the presence of that mode in the system. The mean time to failure of the system is increased through removal of observed failure modes.

Finally, Wolman [6] considers a system in which failures are of two types, which we shall call "inherent" and "assignable cause" following the terminology of Barlow and Scheuer [1]. Each trial may result in exactly one of the following: success, inherent failure, or assignable cause failure. It is assumed that the system originally contains k assignable cause failure modes, and once failure of such a mode is observed the mode is permanently repaired. On each trial, the remaining assignable cause failure modes each occur with known probability q. Under certain "independence" assumptions, Wolman shows how a Markov chain model can be used to obtain the reliability of the system after n trials have been made.

The consideration of several assignable cause modes in the system makes it important to distinguish carefully between the observation of a failure and the repair of a mode. This is of importance, for example, in consistently modeling <u>independence</u> of failure modes. It is also of interest to consider various "repair policies" and their effect upon system reliability after n trials.

In what follows, we consider models encompassing the various features mentioned above. Since the computational difficulty in a particular application may be substantial, some effort is directed toward establishing several methods that can be used for finding the (predicted) reliability of the system after n trials. It should be kept in mind that, throughout the paper, the problem being considered is that of <u>predicting</u>, <u>before testing is undertaken</u>, what the reliability of the system will be after n trials or, from another point of view, predicting the number of trials required to attain a given reliability. Because of its simplicity, we begin with a discussion of a system with a single assignable cause mode, Lloyd and Lipow's model being a particular example (model I). We then extend to a system with k assignable cause modes, assumed to be "equally likely" (model II). Finally, a system with k non-equally likely modes is considered (model III).

The following notation conventions were found to be useful.

(a) Empty sums (that is, summations in which the upper limit on the index is smaller than the lower limit) are taken to be zero.

(b) Empty products are taken to be one.

(c) In some conditional probability statements involving the outcome of a test, given a condition just prior to the test, no specific mention of this "timing" is made. Thus, for example, $P(E \mid E)$ might denote the conditional probability that the system is in state E

immediately after a trial, given that it is in state E just prior to the trial. The context should make the meaning of such symbols clear.

2. Model I: A Single Assignable Cause Mode

Suppose the system under consideration contains two "failure modes", one associated with inherent failures and the other associated with assignable cause failures. (The "inherent failure mode" is imagined to exist for ease of exposition. In particular, it may be taken to be that part of the system other than the assignable cause failure mode.) Assume that on each trial the probability of an inherent failure is q_0, and if the system is unrepaired and an inherent failure does not occur, the probability of an assignable cause failure is q; otherwise, the probability of an assignable cause failure is zero. If we let I_n denote the event "an inherent failure on the n^{th} trial", A_n denote the event "an assignable cause failure on the n^{th} trial", and if we say that the system is in state 0 for the n^{th} trial if it is unrepaired after $n - 1$ trials (denoted by "$E_{0,n-1}$"), while if repaired after $n - 1$ trials the system is in state 1 for the n^{th} trial (denoted by "$E_{1,n-1}$"), the above assumptions can be stated formally as follows:

for $n = 1, 2, 3, \ldots,$

$$P[I_n \cap \bar{A}_n] = P[I_n | E_{1,n-1}]$$

$$= P[I_n | E_{0,n-1}] = q_0$$

$$P[I_n \cap A_n] = P[A_n | E_{1,n-1}] = 0$$

$$P[A_n | \bar{I}_n \cap E_{0,n-1}] = q$$

It follows that the conditional probability of an assignable cause failure on the n^{th} trial, given the system is in state 0, is

$$P[A_n \mid E_{0,\, n-1}] = (1 - q_0)q$$

Before considering a more general repair policy, we digress somewhat to consider the relatively simple repair policy of Lloyd and Lipow; namely, the system is repaired at (i. e., immediately after) the n^{th} trial with probability ϕ if an assignable cause failure occurs at the n^{th} trial; otherwise, it is repaired with probability zero. (This corresponds to the repair policy in which an attempt is made to repair the assignable cause failure mode each time it is observed to fail, these attempts being "identical".)

Since the reliability of the system after n trials, R_n, is given in this case by

$$\begin{aligned} R_n &= \sum_{i=0}^{1} P[\overline{I}_{n+1} \cap \overline{A}_{n+1} \mid E_{i,\, n}] \cdot P(E_{i,\, n}) \\ &= P[\overline{I}_{n+1} \cap \overline{A}_{n+1} \mid E_{0,\, n}] \cdot P(E_{0,\, n}) \\ &\qquad + (1 - P[I_{n+1} \cup A_{n+1} \mid E_{1,\, n}]) \cdot P(\overline{E}_{0,\, n}) \\ &= (1 - q_0)(1 - q)\, P(E_{0,\, n}) + (1 - q_0)(1 - P(E_{0,\, n})) \\ &= (1 - q_0)(1 - q\;\; P(E_{0,\, n})) \qquad (1) \end{aligned}$$

it is of interest to determine $P(E_{0,n})$ for $n = 1, 2, \ldots$. Lloyd and Lipow solve essentially the same problem by considering a difference equation for $P(E_{0,n})$. An alternative is as follows. The one-step transition probability matrix $(p_{ij}^{(1)})$ for this process is easily seen to be

$$(p_{ij}^{(1)}) = \begin{bmatrix} q_0 + (1 - q_0)[(1 - q) + q(1 - \phi)] & (1 - q_0)q\phi \\ 0 & 1 \end{bmatrix}$$

where

$$p_{ij}^{(1)} = P[E_{j, n+1} \mid E_{i, n}]$$

that is, $p_{ij}^{(1)}$ is the probability that the system goes from state i to state j in one trial. The n-step transition probability matrix

$$(p_{ij}^{(n)}) = (p_{ij}^{(1)})^n$$

is thus given by

$$(p_{ij}^{(n)}) = \begin{bmatrix} (1 - g\phi)^n & 1 - (1 - g\phi)^n \\ 0 & 1 \end{bmatrix}$$

where

$$g = (1 - q_0)q$$

This is easily established by observing that each main diagonal element in the product of the two upper triangular matrices is the product of the corresponding main diagonal elements from these matrices and that a product of stochastic matrices is stochastic. If the process starts in state 0 with probability β_0 and if

$$\beta = (\beta_0, 1 - \beta_0)$$

then

$$\beta (p_{ij}^{(n)}) = (P(E_{0,n}), 1 - P(E_{0,n}))$$

so

$$P(E_{0,n}) = \beta_0 (1 - g\phi)^n$$

Substituting this in (1) yields the predicted reliability of the system after n trials,

$$R_n = (1 - q_0)(1 - q\,\beta_0 (1 - g\phi)^n) \tag{2}$$

Note that this reliability is conveniently expressed in a matrix product form by

$$R_n = \beta (p_{ij}^{(n)}) \begin{pmatrix} (1 - q_0)(1 - q) \\ (1 - q_0) \end{pmatrix} \tag{3}$$

The special case

$$q_0 = 1 - \beta_0 = 0$$

results in the expression for R_n derived by Lloyd and Lipow.

A more general repair policy that might be of interest in the present model is as follows. Suppose the probability that the assignable cause failure mode is repaired at its u^{th} failure is b_u; $u = 1, 2, \ldots$. Then

$$P(E_{0,n}) = P[A_n \ \& \ (\text{no repair}) \mid E_{0,n-1}] \cdot P(E_{0,n-1})$$

$$+ P[\overline{A}_n \mid E_{0,n-1}] \cdot P(E_{0,n-1})$$

$$= P(E_{0,n-1}) \Big[\sum_{u=1}^{n} [u^{th} \text{ assignable cause failure at } n^{th} \text{ trial} \mid E_{0,n-1}] \cdot (1 - b_u) + (1 - g) \Big]$$

Now

$$P[u^{th} \text{ assignable cause failure at } n^{th} \text{ trial} \mid E_{0,n-1}]$$

$$= P[u - 1 \text{ assignable cause failures in } n - 1 \text{ trials} \ \& \ A_n \mid E_{0,n-1}]$$

$$= \binom{n-1}{u-1} g^{u-1} (1 - g)^{n-u} \cdot g$$

Let

$$s_n = \sum_{u=1}^{n} \binom{n-1}{u-1} g^u (1-g)^{n-u} (1-b_u) \; ; \; n = 1, 2, \ldots$$

Then

$$P(E_{0,n}) = P(E_{0,n-1}) [s_n + (1-g)]$$

where $P(E_{0,0}) = \beta_0$, the probability the system starts in state 0 .

It follows that

$$P(E_{0,n}) = \beta_0 \prod_{j=1}^{n} (s_j + (1-g)) \; ; \quad n = 0, 1, 2, \ldots$$

Thus, by (1),

$$R_n = (1-q_0)(1 - q\,\beta_0 \prod_{j=1}^{n} (s_j + (1-(1-q_0)q)) \; ; \quad n = 0, 1, 2, \ldots \tag{4}$$

As an example, note that in the case where $b_u \equiv \phi$, (4) reduces to the expression given in (2) , as should be the case. Suppose, for another example, that a repair attempt is made only after the j^{th} assignable cause failure, with probability b of a successful repair. Then R_n is given by (4) , where $b_u = 0$ for $u \neq j$, $b_j = b$. In this case, $s_n = g$ for $n < j$ and $s_n = g(1 - b \binom{n-1}{j-1} g^{j-1} (1-g)^{n-j})$ for $n \geq j$. Then $P(E_{0,n}) = \beta_0$ for $n < j$,

$$P(E_{0,n}) = \beta_0 \prod_{k=j}^{n} \left[1 - b \binom{k-1}{j-1} g^j (1-g)^{k-j} \right]$$

for $n \geq j$. Thus, for example, when $n \geq j$,

$$R_n = (1 - q_0)(1 - q\, \beta_0 \prod_{k=j}^{n} \left[1 - b \binom{k-1}{j-1} g^j (1-g)^{k-j} \right])$$

3. Model II: k Equally Likely Assignable Cause Modes

We now consider the basic model of Wolman [6] in a setting similar to that of Model I in the preceding section, together with some alternatives and generalizations concerning repair policies, initial states of the system, and methods of computing R_n. It is hoped that the present approach clarifies the role of Wolman's "independence" assumptions and corrects several typographical errors in a discussion of Wolman's paper by George Weiss [4].

Suppose the system under consideration is composed of k assignable cause failure modes (hereafter called "type II" modes) and an inherent failure mode (called a "type I" mode), with a series hookup as depicted in Figure 1.

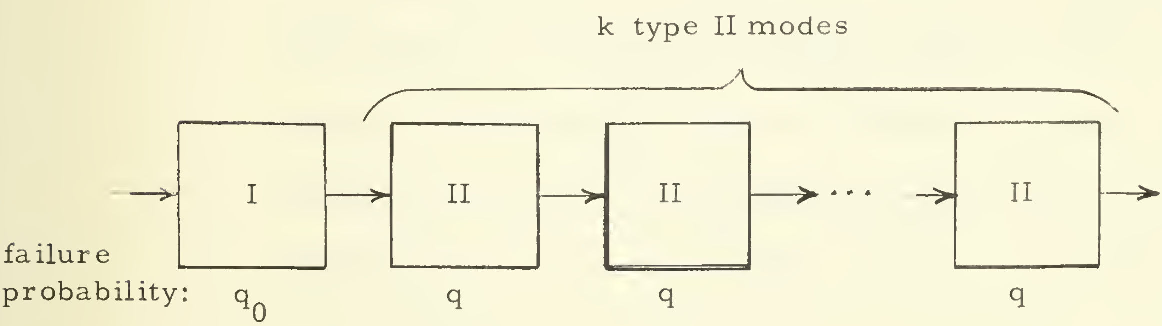

FIGURE I

Suppose "I_n" denotes, as before, the event that an inherent failure occurs on the n^{th} trial, while "A_n" denotes the event that at least one assignable cause failure occurs on the n^{th} trial. We make the following assumptions: on a given trial (i.e., for any n, $n = 1, 2, \ldots$),

1. $$P[I_n] = P[I_n \cap \overline{A}_n] = q_0$$

   (the inherent failure probability is constant).

2. If the j^{th} type II mode is unrepaired prior to the n^{th} trial,

   $$P[j^{th} \text{ II fails} \mid I_n] = 0 ,$$

   $$P[j^{th} \text{ II fails} \mid \overline{I}_n] = q = 1 - p$$

   (assignable cause failures in unrepaired type II modes are equally likely and may occur only if an inherent failure does not occur).

3. The type II modes are independent in the sense that given that there are several type II modes which are unrepaired prior to a trial and given that an inherent failure does not occur on the trial, the probability that these type II modes all fail on the trial is the product of the corresponding probabilities q.

We say that the system is in state i $(i = 0, 1, \ldots, k)$ for the n^{th} trial if i type II modes have been (permanently) repaired after $n - 1$ trials. Let "$E_{i,\, n-1}$" denote the event that the system is in

state i, assumption (3) above implies that on the n^{th} trial

$$P[\overline{A}_n \mid E_{i, n-1} \cap \overline{I}_n] = p^{k-i}$$

Let us first consider the repair policy which leads to Wolman's model.

<u>If, on a given trial, at least one type II mode fails, exactly one type II mode is repaired.</u>

Under this repair policy, it is relatively straightforward to compute R_n by any one of several approaches. These approaches have in common a consideration of the transition probabilities of the system. The $k+1 \times k+1$ matrix of one-step transition probabilities has a simple form in the present case. In fact, on any trial, for $i = 0, 1, 2, \ldots, k$,

$$p_{ii}^{(1)} = P[E_{i,n} \mid E_{i,n-1}] = P[\overline{A}_n \mid E_{i,n-1}]$$

$$= P[\overline{A}_n \mid E_{i,n-1} \cap \overline{I}_n] \cdot P[\overline{I}_n]$$

$$+ P[\overline{A}_n \mid E_{i,n-1} \cap I_n] \cdot P[I_n]$$

$$= p^{k-i}(1 - q_0) + q_0 \tag{5}$$

$$p_{i,i+1}^{(1)} = P[E_{i+1,n} \mid E_{i,n-1}] = P[A_n \mid E_{i,n-1}]$$

$$= 1 - p_{ii}^{(1)} = (1 - q_0)(1 - p^{k-i}) \tag{6}$$

Thus,

$$(p_{ij}^{(1)}) = \begin{bmatrix} q_0+(1-q_0)p^k & (1-q_0)(1-p^k) & 0 & 0 & \cdots & 0 \\ 0 & q_0+(1-q_0)p^{k-1} & (1-q_0)(1-p^{k-1}) & 0 & \cdots & 0 \\ 0 & 0 & q_0+(1-q_0)p^{k-2} & (1-q_0)(1-p^{k-2}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

If $\beta = (\beta_0, \beta_1, \beta_2, \ldots, \beta_k)$ is the distribution of initial states of the system, that is, if

$$\beta_i = P[\text{ system starts in state } i]; \; i = 0, 1, \ldots, k$$

then $\beta(p_{ij}^{(n)})$ is the distribution of states occupied by the system after n trials. The matrix form (3) of the reliability given in (1) is easily extended to the present case, so that the reliability after n trials is given by

$$R_n = \beta(p_{ij}^{(n)}) \begin{pmatrix} p^k \\ p^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} (1 - q_0) \tag{7}$$

The determination of $(p_{ij}^{(n)})$ is not as simple in the present case as it was for Model I. Wolman considers an approach based upon a "spectral decomposition" of $(p_{ij}^{(1)})$, as follows: the characteristic roots of the positive definite matrix $(p_{ij}^{(1)})$ are simply its diagonal elements,

$$\lambda_i = q_0 + (1 - q_0) p^{k - i} ; \; i = 0, 1, \ldots, k$$

Thus, if X is a $k + 1 \times k + 1$ matrix whose rows are linearly independent characteristic vectors of P and Λ is the diagonal matrix of corresponding λ_i's, then

$$(p_{ij}^{(n)}) = X^{-1} \Lambda^n X ; \; n = 1, 2, \ldots$$

Thus, once X^{-1} and X have been determined (possibly more than a routine task), $(p_{ij}^{(n)})$ is easily computed for all n since

$$\Lambda^n = \begin{bmatrix} \lambda_0^n & 0 & \cdots & 0 \\ 0 & \lambda_1^n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_k^n \end{bmatrix}$$

An alternate approach to the determination of $(p_{ij}^{(n)})$ has been suggested by Weiss [4]. In this approach, the elements in each row of $(p_{ij}^{(n)})$ are computed through the use of certain generating functions.

We illustrate this method for the first row of $(p_{ij}^{(n)})$. Let

$$G_{0i}(s) = \sum_{n=0}^{\infty} s^n p_{0i}^{(n)} \;;\quad s \in (0, 1], \; i = 0, 1, \ldots, k$$

Note that, since $p_{00}^{(n)} = (p_{00}^{(1)})^n$,

$$G_{00}(s) = \sum_{n=0}^{\infty} s^n p_{00}^{(n)} = \sum_{n=0}^{\infty} (s[q_0 + (1 - q_0) p^k])^n$$

$$= \frac{1}{1 - p_{00}^{(1)} s}$$

Also,

$$p_{0i}^{(n)} = p_{0,i-1}^{(n-1)} \cdot p_{i-1,i}^{(1)} + p_{0i}^{(n-1)} p_{ii}^{(1)} \;;\quad i = 1, 2, \ldots, k \qquad (8)$$

Thus,

$$G_{0i}(s) = \sum_{n=0}^{\infty} s^n \left[p_{0,i-1}^{(n-1)} p_{i-1,i}^{(1)} + p_{0i}^{(n-1)} p_{ii}^{(1)} \right]$$

$$= p_{i-1,i}^{(1)} s \sum_{n=0}^{\infty} s^n p_{0,i-1}^{(n)} + p_{ii}^{(1)} s \sum_{n=0}^{\infty} s^n p_{0i}^{(n)}$$

$$= p_{i-1,i}^{(1)} s G_{0,i-1}(s) + p_{ii}^{(1)} s G_{0i}(s) \;;\quad i = 1, 2, \ldots, k$$

Then

$$G_{0i}(s) \left[1 - p_{ii}^{(1)} s \right] = p_{i-1,i}^{(1)} s G_{0,i-1}(s)$$

and

$$G_{0i}(s) = \frac{s\, p_{i-1,\,i}^{(1)}\, G_{0,\,i-1}(s)}{1 - p_{ii}^{(1)}\, s} = \frac{s\, p_{i-1,\,i}^{(1)}}{1 - p_{ii}^{(1)}\, s} \cdot \frac{s\, p_{i-2,\,i-1}^{(1)}\, G_{0,\,i-2}(s)}{1 - p_{i-1,\,i-1}^{(1)}\, s}$$

$$\vdots$$

$$= \frac{s^{i}\, p_{01}^{(1)}\, p_{12}^{(1)} \cdots p_{i-1,\,i}^{(1)}}{(1 - p_{11}^{(1)}\, s)\,(1 - p_{22}^{(1)}\, s) \cdots (1 - p_{ii}^{(1)}\, s)} \cdot G_{00}(s)$$

$$= \left(s^{i} \prod_{j=1}^{i} p_{j-1,\,j}^{(1)} \right) \frac{1}{\prod_{j=0}^{i} (1 - p_{jj}^{(1)}\, s)} \quad ; \; i = 0, 1, \ldots, k \qquad (9)$$

Using partial fractions, the second factor in (9) can be expressed as

$$\frac{(p_{00}^{(1)})^{i}}{\prod_{\substack{j=0 \\ j\neq 0}}^{i} (p_{00}^{(1)} - p_{jj}^{(1)})} \cdot \frac{1}{1 - p_{00}^{(1)}\, s} + \frac{(p_{11}^{(1)})^{i}}{\prod_{\substack{j=0 \\ j\neq 1}}^{i} (p_{11}^{(1)} - p_{jj}^{(1)})}$$

$$\cdot \frac{1}{1 - p_{11}^{(1)}\, s} + \cdots + \frac{(p_{ii}^{(1)})^{i}}{\prod_{\substack{j=0 \\ j\neq i}}^{i} (p_{ii}^{(1)} - p_{jj}^{(1)})} \cdot \frac{1}{1 - p_{ii}^{(1)}\, s}$$

so that

$$\sum_{n=0}^{\infty} s^n p_{0i}^{(n)} = \left(s^i \prod_{j=1}^{i} p_{j-1,j}^{(1)} \right) \left[\frac{(p_{00}^{(1)})^i}{\prod_{j \neq 0} (p_{00}^{(1)} - p_{jj}^{(1)})} \right.$$

$$\left. \cdot \left(\sum_{m=0}^{\infty} (p_{00}^{(1)} s)^m \right) + \ldots + \frac{(p_{ii}^{(1)})^i}{\prod_{j \neq i} (p_{ii}^{(1)} - p_{jj}^{(1)})} \sum_{m=0}^{\infty} (p_{ii}^{(1)} s)^m \right]$$

By comparing coefficients of the corresponding powers of s in this expression, we find that

$$p_{0i}^{(n)} = \begin{cases} 0 & \text{for } n < i \\ \left(\prod_{j=1}^{i} p_{j-1,j}^{(1)} \right) \sum_{m=0}^{i} \dfrac{(p_{mm}^{(1)})^n}{\prod_{j \neq m} (p_{mm}^{(1)} - p_{jj}^{(1)})} & \text{for } n \geq i \end{cases}$$

The other rows of $(p_{ij}^{(n)})$ are similarly determined, and R_n is computed by (7).

An alternate approach under the present repair policy that might be of some interest is as follows: note that for fixed n $(n = 0, 1, 2, \ldots)$ and i $(i = 0, 1, \ldots, k)$

$$p_{i,\, i+j}^{(n)} \; ; \; j = 0, 1, \ldots, k - i$$

is a mass function, since $(p_{ij}^{(n)})$ is a stochastic matrix. Let $J_i^{(n)}$ denote the random variables having these distributions. Since

$$R_n = (1 - q_0)\, \beta\, (p_{ij}^{(n)}) \begin{pmatrix} p^k \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

the major task in computing R_n is to find the elements of the vector

$$\left(p_{ij}^{(n)}\right) \begin{pmatrix} p^k \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{k} p_{0j}^{(n)}\, p^{k-j} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=0}^{k} p_{kj}^{(n)}\, p^{k-j} \end{pmatrix}$$

Consider the first component:

$$\sum_{j=0}^{k} p_{0j}^{(n)} p^{k-j} = p^k \sum_{j=0}^{k} p_{0j}^{(n)} \left(\frac{1}{p}\right)^j$$

$$= p^k \, E\left(\left(\frac{1}{p}\right)^{J_0^{(n)}} \right)$$

$$= p^k \, \psi_{J_0^{(n)}}\left(\frac{1}{p}\right)$$

where

$$\psi_{J_0^{(n)}}(t) = E\left(t^{J_0^{(n)}}\right) ; \quad t \in (0, 1]$$

is the probability generating function of the random variable $J_0^{(n)}$.

A recursion relation for the generating functions $\psi_{J_0^{(n)}}(t)$ can be obtained using the one-step transition probabilities given in (5) and (6) together with equation (8). Thus,

$$\psi_{J_0^{(m+1)}}(t) = \sum_{i=0}^{k} t^i \Big[p_{0i}^{(m)} \left[q_0 + (1 - q_0)\, p^{k-i} \right]$$

$$+ \; p_{0,\,i-1}^{(m)} \left[(1 - q_0)(1 - p^{k-i+1}) \right] \Big]$$

$$= q_0 \sum_{i=0}^{k} t^i \, p_{0i}^{(n)} + (1 - q_0)\, p^k \sum_{i=0}^{k} \left(\frac{t}{p}\right)^i p_{0i}^{(m)}$$

$$+ \; t \sum_{i=0}^{k} t^{i-1} \, p_{0,\,i-1}^{(m)} \left[(1 - q_0)(1 - p^{k-(i-1)}) \right]$$

$$= q_0 \, \psi_{J_0^{(m)}}(t) + (1 - q_0)\, p^k \, \psi_{J_0^{(m)}}\left(\frac{t}{p}\right)$$

$$+ \; (1 - q_0)\, t \left[\psi_{J_0^{(m)}}(t) - p^k \, \psi_{J_0^{(m)}}\left(\frac{t}{p}\right) \right]$$

Starting from

$$\psi_{J_0^{(0)}}(t) = E\left(t^{J_0^{(0)}}\right) = 1$$

one can, in theory, compute $\psi_{J_0^{(m)}}(t)$ for successively higher values of m. Unfortunately, this computation becomes tedious for large m values.

In matrix form, the vector of probability generating functions $\psi_{J_i^{(n)}}(t)$ is given by

$$(p_{ij}^{(n)})\, T = \begin{bmatrix} \psi_{J_0^{(n)}}(t) \\ \cdot \\ \cdot \\ \cdot \\ \psi_{J_k^{(n)}}(t) \end{bmatrix}$$

where

$$T = \begin{bmatrix} 1 \\ t \\ t^2 \\ \cdot \\ \cdot \\ \cdot \\ t^k \end{bmatrix}$$

Since a convex combination of probability generating functions is a probability generating function,

$$\beta\, (p_{ij}^{(n)})\, T = \psi_n(t)$$

is a probability generating function. (In fact, ψ_n is the probability generating function of the random variable denoting the state occupied after n trials.) It then follows that

$$R_n = (1 - q_0)\, p^k\, \psi_n\left(\frac{1}{p}\right)$$

Remark. It is easy to incorporate into this consideration a slightly more general repair policy, as follows.

A type II mode is repaired with probability ϕ on a given trial, given at least one such mode fails on that trial.

In this case, we replace the original one-step transition probability matrix by one in which, for $i = 0, 1, \ldots, k;\ n = 1, 2, \ldots$

$$\begin{aligned} p_{ii}^{(1)} &= P[\bar{A}_n \mid E_{i,n-1}] + P[A_n \ \&\ \text{no repair} \mid E_{i,n-1}] \\ &= q_0 + (1 - q_0)p^{k-i} + (1 - q_0)(1 - p^{k-i})(1 - \phi) \end{aligned}$$

$$\begin{aligned} p_{i,i+1}^{(1)} &= P[A_n \ \&\ \text{a repair}] \\ &= (1 - q_0)(1 - p^{k-i})\,\phi \end{aligned}$$

Of the three approaches to computing R_n discussed for the present repair policy, it appears that none is clearly "best" under all circumstances. A choice of which to use would probably depend, at least in part, upon factors such as computer availability, size of the matrix $(p_{ij}^{(1)})$, and the number of R_n's to be computed. It is conceivable that, in some circumstances, the approach consisting of simply multiplying $(p_{ij}^{(1)})$ by itself the required number of times is worth consideration.

Let us next consider the following repair policy: at each trial all type II modes that fail are repaired. In this case, the matrix $(p_{ij}^{(1)})$ has elements such that

$$p_{mi}^{(1)} = 0 \qquad \text{for} \quad m > i \tag{10}$$

$$p_{ii}^{(1)} = P[\bar{A}_n | E_{i, n-1}] = q_0 + (1 - q_0) p^{k-i} \; ; \; i = 0, 1, \ldots, k \tag{11}$$

$$p_{i, i+j}^{(1)} = (1 - q_0) \binom{k-i}{j} p^{k-i-j} q^j \; ; \quad \begin{array}{l} i = 0, 1, \ldots, k \\ j = 1, 2, \ldots, k - i \end{array} \tag{12}$$

With this $(p_{ij}^{(1)})$ matrix, one could compute R_n using the spectral decomposition approach discussed above. The generating function approaches can also be used, although the computation becomes more involved. This is because simple recurrence relations of the form (8) under the former repair policy must now be replaced with relations of the form

$$p_{0i}^{(n)} = p_{0i}^{(n-1)} p_{ii}^{(1)} + p_{0, i-1}^{(n-1)} p_{i-1, i}^{(1)} + \ldots + p_{00}^{(n-1)} p_{0i}^{(1)}$$

Again, the "best" method to use to calculate R_n would probably depend upon the particular circumstances involved.

An interesting modification of the repair policy in which all type II failures at each stage are corrected is as follows.

At each stage, each observed type II failure is corrected with probability ϕ, independent of other failures and the number of failures.

In this case, elements of $(p_{ij}^{(1)})$ are obtained by replacing q with ϕq in (10), (11), and (12). This rather intuitively obvious statement may be more rigorously established as follows:

$$p_{ii}^{(1)} = P[\bar{A}_n \mid E_{i,n-1}] + (1-\phi)$$

$$\cdot P[\text{ one type II mode failure} \mid E_{i,n-1}] + \ldots$$

$$+ (1-\phi)^{k-i} P[\text{ k-i type II mode failures} \mid E_{i,n-1}]$$

$$= q_0 + (1-q_0) \left[\sum_{j=0}^{k-i} p^{k-i-j} \binom{k-i}{j} \right.$$

$$\left. \cdot \{(1-\phi)q\}^j \right]$$

$$= q_0 + (1-q_0)[p + (1-\phi)q]^{k-i}$$

$$= q_0 + (1-q_0)[1 - \phi q]^{k-i}$$

$$p_{i,i+1}^{(1)} = P[\text{ one type II mode failure \& one repair} \mid E_{i,n-1}] + \ldots$$

$$+ P[\text{k-i type II mode failures \& one repair} \mid E_{i,n-1}]$$

$$= (1-q_0)\phi \sum_{j=1}^{k-i} \binom{k-i}{j} p^{k-i-j} q^j \binom{j}{1} (1-\phi)^{j-1} \qquad (13)$$

noting that

$$\frac{d}{d\phi}\, p_{ii}^{(1)} = (1 - q_0) \sum_{j=1}^{k-i} p^{k-i-j} \binom{k-i}{j} j$$

$$\cdot \{ (1-\phi) q \}^{j-1} (-q)$$

$$= (1 - q_0)(k - i) [p + (1 - \phi) q]^{k-i-1} (-q)$$

it follows that (13) is equivalent to

$$p_{i,\, i+1}^{(1)} = (1 - q_0)\, \phi\, q\, (k - i) [1 - \phi q]^{k-i-1}$$

Similarly, it follows that

$$p_{i,\, i+j}^{(1)} = (1 - q_0) \frac{\phi^j}{j!} \sum_{m=j}^{k-i} \binom{k-i}{m} p^{k-i-m} q^m$$

$$\cdot\ m (m-1) \ldots (m - j + 1) (1 - \phi)^{m-j}$$

$$= (1 - q_0) \frac{\phi^j}{j!} q^j (k - i)(k - i - 1) \ldots (k - i - j + 1)$$

$$\cdot\ [p + (1 - \phi) q]^{k-i-j}$$

$$= (1 - q_0)(\phi q)^j \binom{k-i}{j} (1 - \phi q)^{k-i-j}$$

for $i = 0, 1, \ldots, k$; $j = 1, 2, \ldots, k - i$.

4. Model III: k Assignable Cause Modes, Not Necessarily Equally Likely

Markov chain models for the case in which type II failures are not equally likely are equivalent to models in which type II failures are equally likely but the repair policy does not treat the failure modes "uniformly". In what follows, we shall concentrate mostly upon systems assumed to have two type II modes ($k = 2$) in series (see Figure 2).

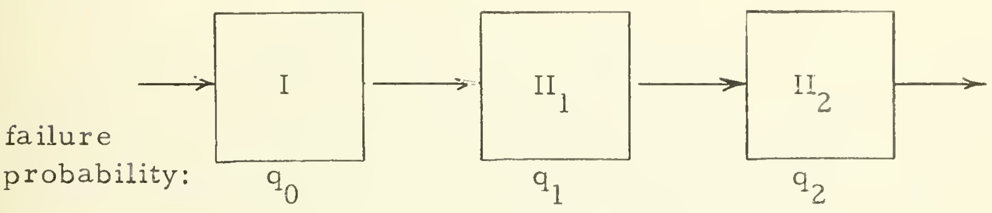

FIGURE 2

One aspect of Model III which is different from the previous models is the identification of "states" of the system. For example, suppose $k = 2$. We may define states as follows:

State	Condition
0	no type II mode repaired
1	mode II_1 repaired
2	mode II_2 repaired
3	both type II modes repaired

In general, with k type II modes, there are 2^k states of this nature. Once the corresponding $(p_{ij}^{(1)})$ matrix has been obtained, the value of R_n is given by

$$R_n = (1 - q_0)\, \beta\, (p_{ij}^{(n)})\, P \tag{14}$$

where the j^{th} component of the vector P is the conditional probability that no type II failures occur, given the system is in state j and a type I failure does not occur.

Let us assume that at each stage all type II failures are (permanently) repaired, and consider two example models, each depending upon conditions imposed on the occurrence of type II failures. Extensions to other values of k and other repair policies can be made as in the models considered previously. Consider first the case in which it is assumed that II_2 can fail on a given trial only if both II_1 and I do not fail. More precisely, assume that

$$P[\text{ I fails }] = q_0$$

$$P[\text{ II}_1 \text{ fails} \mid \text{no I failure}] = q_1$$

$$P[\text{ II}_2 \text{ fails} \mid \text{no I failure \& no II}_1 \text{ failure}] = q_2$$

$$P[\text{ II}_2 \text{ fails} \mid \text{I or II}_1 \text{ fails}] = P[\text{ II}_1 \text{ fails} \mid \text{I fails}] = 0$$

Under these "exclusive occurrence" assumptions, the one-step transition matrix is of the form

$$
(p_{ij}^{(1)}) = \begin{bmatrix} p_{00}^{(1)} & p_{01}^{(1)} & p_{02}^{(1)} & p_{03}^{(1)} \\ 0 & p_{11}^{(1)} & 0 & p_{13}^{(1)} \\ 0 & 0 & p_{22}^{(1)} & p_{23}^{(1)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}
$$

where

$$p_{00}^{(1)} = q_0 + \prod_{i=0}^{2} (1 - q_i)$$

$$p_{01}^{(1)} = (1 - q_0)\, q_1$$

$$p_{02}^{(1)} = q_2\, (1 - q_0)\, (1 - q_1)$$

$$p_{03}^{(1)} = 0$$

$$p_{11}^{(1)} = q_0 + (1 - q_0)\, (1 - q_2)$$

$$p_{13}^{(1)} = (1 - q_0)\, q_2$$

$$p_{22}^{(1)} = q_0 + (1 - q_0)\, (1 - q_1)$$

$$p_{23}^{(1)} = (1 - q_0)\, q_1$$

Note: This process can be viewed as a walk on the plane or as a network, as is suggested in Figure 3.

FIGURE 3

Next, consider the case in which the type II modes are independent. Let

$$P[\text{ I fails }] = q_0$$

$$P[\text{ II}_1 \text{ fails} \mid \text{no I failure }] = q_1$$

$$P[\text{ II}_2 \text{ fails} \mid \text{no I failure }] = q_2$$

$$P[\text{II}_1 \text{ fails \& II}_2 \text{ fails} \mid \text{no I failure }] = q_1 \; q_2$$

assuming neither type II mode is repaired. In this case, the matrix $(p_{ij}^{(1)})$ is of the form given in (15), with entries as before except that

$$p_{01}^{(1)} = (1 - q_0)(1 - q_2) q_1$$

$$p_{03}^{(1)} = (1 - q_0) q_1 q_2$$

(The fact that $p_{03}^{(1)}$ is no longer zero allows passage along the dotted paths of Figure 3.) Also, note that in this case the vector P of (14) is given by

$$P = \begin{bmatrix} (1 - q_1)(1 - q_2) \\ (1 - q_2) \\ (1 - q_1) \\ 1 \end{bmatrix}$$

In both of these examples and, in general for other values of k, the computation of $(p_{ij}^{(n)})$ in Model III can be accomplished using the generating function approach or the spectral decomposition approach. The probability generating function method no longer applies, however, because of the form of the vector P.

5. Summary

In viewing reliability growth prediction models as Markov chains, it is seen that the computation of the reliability after n trials and possible associated repairs, R_n, may be accomplished with any of several different methods. A class of models is considered which accommodates variations in several important factors such as the interdependencies of assignable cause failure modes, inclusion of an inherent failure mode, the repair policy, and the distribution of initial states of the system.

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13. ABSTRACT

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